

# New results on path-decompositions and their down-links

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## Abstract

In [3] the concept of *down-link* from a  $(K_v, \Gamma)$ -design  $\mathcal{B}$  to a  $(K_n, \Gamma')$ -design  $\mathcal{B}'$  has been introduced. In the present paper the spectrum problems for  $\Gamma' = P_4$  are studied. General results on the existence of path-decompositions and embeddings between path-decompositions playing a fundamental role for the construction of down-links are also presented.

**Keywords:**  $(K_v, \Gamma)$ -design; down-link; embedding.

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## 1 Introduction

Suppose  $\Gamma \leq K$  to be a subgraph of  $K$ . A  $(K, \Gamma)$ -*design*, or  $\Gamma$ -*decomposition* of  $K$ , is a set of graphs isomorphic to  $\Gamma$  whose edges partition the edge set of  $K$ . Given a graph  $\Gamma$ , the problem of determining the existence of  $(K_v, \Gamma)$ -designs, also called  $\Gamma$ -*designs of order  $v$* , where  $K_v$  is the complete graph on  $v$  vertices, has been extensively studied; see the surveys [4, 5]. In [3] we proposed the following definition.

**Definition 1.1.** *Given a  $(K, \Gamma)$ -design  $\mathcal{B}$  and a  $(K', \Gamma')$ -design  $\mathcal{B}'$  with  $\Gamma' \leq \Gamma$ , a down-link from  $\mathcal{B}$  to  $\mathcal{B}'$  is a function  $f : \mathcal{B} \rightarrow \mathcal{B}'$  such that  $f(B) \leq B$ , for any  $B \in \mathcal{B}$ .*

When such a function  $f$  exists, we say that it is possible to *down-link*  $\mathcal{B}$  to  $\mathcal{B}'$ .

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As seen in [3], down-links are closely related to metamorphoses [8], their generalizations [9] and embeddings [11]. In close analogy to embeddings, we introduced spectrum problems about down-links:

- (I) For each admissible  $v$ , determine the set  $\mathcal{L}_1\Gamma(v)$  of all integers  $n$  such that there exists *some*  $\Gamma$ -design of order  $v$  down-linked to a  $\Gamma'$ -design of order  $n$ .
- (II) For each admissible  $v$ , determine the set  $\mathcal{L}_2\Gamma(v)$  of all integers  $n$  such that *every*  $\Gamma$ -design of order  $v$  can be down-linked to a  $\Gamma'$ -design of order  $n$ .

In [3, Proposition 3.2], we proved that for any  $v$  such that there exists a  $(K_v, \Gamma)$ -design and any  $\Gamma' \leq \Gamma$ , the sets  $\mathcal{L}_1\Gamma(v)$  and  $\mathcal{L}_2\Gamma(v)$  are always non-empty. In the same paper the case  $\Gamma' = P_3$  has been investigated in detail. Here we shall deal with the case  $\Gamma' = P_4$ . In order to get results about down-links to  $P_4$ -designs, we shall first study path-designs and their embeddings. More precisely, in Section 2 we determine sufficient conditions for the existence of  $P_4$ -decompositions of any graph  $\Gamma$  and  $P_k$ -decompositions of complete bipartite graphs. In Section 3, applying the results of Section 2, we are able to prove the existence of embeddings and down-links between path-designs. Section 4 is devoted to the cases of cycle systems and path-designs, with general theorems and directed constructions.

Throughout this paper the following standard notations will be used; see also [7]. For any graph  $\Gamma$ , write  $V(\Gamma)$  for the set of its vertices and  $E(\Gamma)$  for the set of its edges. If  $\mathcal{B}$  is a collection of graphs, by  $V(\mathcal{B})$  we will mean the set of the vertices of all its elements. By  $t\Gamma$  we shall denote the disjoint union of  $t$  copies of graphs all isomorphic to  $\Gamma$ . As usual,  $P_k = [a_1, \dots, a_k]$  is the path with  $k - 1$  edges and  $C_k = (a_1, \dots, a_k)$ ,  $k \geq 3$ , is the cycle of length  $k$ . Also,  $K_{m,n}$  is the complete bipartite graph with parts of size  $m$  and  $n$ . When we focus on the actual parts  $X$  and  $Y$ ,  $K_{X,Y}$  will be written.

## 2 Existence of some path-designs

In this section we present new results on the existence of path decompositions. Recall that a  $(K_n, P_k)$ -design exists if, and only if,  $n(n - 1) \equiv 0 \pmod{2(k - 1)}$ ; see [13].

**Proposition 2.1.** *Let  $k$  be an even integer. For  $x = k - 2$ ,  $k$  the complete bipartite graph  $K_{k-1,x}$  admits a  $P_k$ -decomposition.*

*Proof.* Consider the bipartite graph  $K_{A,I}$  where  $A = \{a_1, \dots, a_{k-1}\}$  and  $I = \{1, \dots, x\}$  with  $x = k - 2, k$ .

Let  $U^t = (1, \dots, 1)$  be an  $\frac{x}{2}$ -tuple. Set  $P_1^t = (1, \dots, \frac{x}{2})$  and for  $i = 1, \dots, \frac{x}{2}$ ,

$P_i^t = (i, i+1, \dots, \frac{x}{2}, 1, 2, \dots, i-1)$ ,  $\overline{P}_i^t = P_i^t + \frac{x}{2}U$ ,  $A_i = a_iU$ ,  $\overline{A}_i = a_{i+\frac{k}{2}}U$ .  
If  $k \equiv 0 \pmod{4}$ , consider the  $\frac{x}{2} \times k$  matrices

$$M = (P_1 \ A_1 \ \overline{P}_1 \ A_2 \ P_2 \ A_3 \ \overline{P}_2 \dots P_i \ A_{2i-1} \ \overline{P}_i \ A_{2i} \dots P_{\frac{k}{4}} \ A_{\frac{k-2}{2}} \ \overline{P}_{\frac{k}{4}} \ A_{\frac{k}{2}})$$

$$\overline{M} = (\overline{P}_1 \ \overline{A}_1 \ P_1 \ \overline{A}_2 \ \overline{P}_2 \ \overline{A}_3 \ P_2 \dots \overline{P}_i \ \overline{A}_{2i-1} \ P_i \ \overline{A}_{2i} \dots \overline{P}_{\frac{k}{4}} \ \overline{A}_{\frac{k-2}{2}} \ P_{\frac{k}{4}} \ A_{\frac{k}{2}}).$$

If  $k \equiv 2 \pmod{4}$ , consider the  $\frac{x}{2} \times k$  matrices

$$M = (P_1 \ A_1 \ \overline{P}_1 \ A_2 \ P_2 \ A_3 \ \overline{P}_2 \dots P_i \ A_{2i-1} \ \overline{P}_i \ A_{2i} \dots P_{\frac{k+2}{4}} \ A_{\frac{k}{2}})$$

$$\overline{M} = (\overline{P}_1 \ \overline{A}_1 \ P_1 \ \overline{A}_2 \ \overline{P}_2 \ \overline{A}_3 \ P_2 \dots \overline{P}_i \ \overline{A}_{2i-1} \ P_i \ \overline{A}_{2i} \dots \overline{P}_{\frac{k+2}{4}} \ A_{\frac{k}{2}}).$$

In either case, the rows of  $M$  and  $\overline{M}$ , taken together, are the  $x$  paths of a  $P_k$ -decomposition of  $K_{A,I}$ .  $\square$

**Theorem 2.2.** *Let  $\Gamma$  be a graph with at least two vertices of degree  $|V(\Gamma)|-1$ . Then  $\Gamma$  admits a  $P_4$ -decomposition if, and only if,  $|E(\Gamma)| \equiv 0 \pmod{3}$ . If  $|E(\Gamma)| \equiv 1, 2 \pmod{3}$ , then  $\Gamma$  can be partitioned into a  $P_4$ -decomposition together with one or two (possibly connected) edges, respectively.*

*Proof.* The condition is obviously necessary. For sufficiency, let  $\alpha$  and  $\beta$  be two vertices of degree  $|V(\Gamma)|-1$ . Delete  $\alpha$  and  $\beta$  in  $\Gamma$ , as to obtain a graph  $G$ . Let  $G'$  be a maximal  $P_4$ -decomposable subgraph of  $G$  and remove from  $G$  the edges of  $G'$ , determining a new graph  $G''$ . In general,  $G''$  is not connected and its connected components are either isolated vertices or stars or cycles of length 3; call  $\mathcal{I}$ ,  $\mathcal{S}$  and  $\mathcal{C}$  their (possibly empty) sets. Let  $\Gamma'$  be the graph obtained removing the edges of  $G'$  from  $\Gamma$ . Clearly,  $|E(\Gamma)| \equiv 0 \pmod{3}$  implies  $|E(\Gamma')| \equiv 0 \pmod{3}$ ; thus it remains to show that  $E(\Gamma')$  is  $P_4$ -decomposable. Obviously  $\alpha$  and  $\beta$  are of degree  $|V(\Gamma)|-1$  also in  $\Gamma'$ . Let  $A = \{\alpha, \beta\}$  and consider the following decomposition  $\Gamma' = K_{A \cup \mathcal{I}} \cup K_{A, \mathcal{I} \cup (\mathcal{C} \cup K_{A, V(\mathcal{C})}) \cup (\mathcal{S} \cup K_{A, V(\mathcal{S})})$ . We begin by providing, separately,  $P_4$ -decompositions of  $K_{A, \mathcal{I}}$ ,  $\mathcal{C} \cup K_{A, V(\mathcal{C})}$  and  $\mathcal{S} \cup K_{A, V(\mathcal{S})}$ .

*i)* It is easy to see that for any 3-subset of  $\mathcal{I}$ , say  $H_3$ , the graph  $K_{A, H_3}$  has a  $P_4$ -decomposition. Thus, depending on the congruence class modulo 3 of  $|\mathcal{I}|$ ,  $K_{A, \mathcal{I}}$  can be partitioned into a  $P_4$ -decomposition together with the following possible remnants.

$(i_1)$ $ \mathcal{I}  \equiv 0 \pmod{3}$	$(i_2)$ $ \mathcal{I}  \equiv 1 \pmod{3}$	$(i_3)$ $ \mathcal{I}  \equiv 2 \pmod{3}$
the set $\emptyset$	the path $[\alpha, h, \beta]$ with $h \in \mathcal{I}$	the cycle $(h_1, \alpha, h_2, \beta)$ with $h_1, h_2 \in \mathcal{I}$

Table 1: Case  $i$ .

ii) For any 3-cycle  $C \in \mathcal{C}$ , the graph  $C \cup K_{A,V(C)}$  has a  $P_4$ -decomposition. Thus,  $\mathcal{C} \cup K_{A,V(C)}$  also admits a  $P_4$ -decomposition.

iii) It is not difficult to see that, for any star  $S_c \in \mathcal{S}$  of center  $c$ , the graph  $S_c \cup K_{A,V(S_c)}$  has a partition into a  $P_4$ -decomposition together with either the path  $[\alpha, c, \beta]$  or the graph  $(\alpha, c, \beta, v) \cup [c, v]$ , where  $v$  is any external vertex, depending on whether the number of vertices of  $S_c$  is odd or even. Let  $\mathcal{S}_1$  (respectively  $\mathcal{S}_2$ ) be the set of stars with an odd (even) number of vertices. For any three stars of  $\mathcal{S}_1$  ( $\mathcal{S}_2$ ) the remnants give  $P_4$ -decomposable graphs. So  $\mathcal{S}_1 \cup K_{A,V(\mathcal{S}_1)}$ , as well as  $\mathcal{S}_2 \cup K_{A,V(\mathcal{S}_2)}$ , can be partitioned into a  $P_4$ -decomposition together with the possible remnants outlined in Tables 2 and 3.

$(iii_{11})$ $ \mathcal{S}_1  \equiv 0 \pmod{3}$	$(iii_{12})$ $ \mathcal{S}_1  \equiv 1 \pmod{3}$	$(iii_{13})$ $ \mathcal{S}_1  \equiv 2 \pmod{3}$
$\emptyset$	the path $[\alpha, c, \beta]$ where $c$ is the center of a star	the cycle $(c_1, \alpha, c_2, \beta)$ where $c_1, c_2$ are centers of two stars

Table 2: Case  $iii_1$ :  $\mathcal{S}_1 \cup K_{A,V(\mathcal{S}_1)}$ .

$(iii_{21})$ $ \mathcal{S}_2  \equiv 0 \pmod{3}$	$(iii_{22})$ $ \mathcal{S}_2  \equiv 1 \pmod{3}$	$(iii_{23})$ $ \mathcal{S}_2  \equiv 2 \pmod{3}$
$\emptyset$	the graph $(\alpha, c, \beta, v) \cup [c, v]$ where $c$ is the center and $v$ is an external vertex of a star	the graph $\bigcup_{i=1}^2 (\alpha, c_i, \beta, v_i) \cup [c_i, v_i]$ where $c_1, c_2$ are centers and $v_1, v_2$ are external vertices of two stars

Table 3: Case  $iii_2$ :  $\mathcal{S}_2 \cup K_{A,V(\mathcal{S}_2)}$ .

The remnants from  $i)$ ,  $iii_1)$  and  $iii_2)$  together with the edge  $[\alpha, \beta]$  can be combined in 27 different ways to obtain 27 connected graphs with  $t$  edges. It is a routine to check that we have exactly 9 cases with  $t \equiv i \pmod{3}$ , for  $i = 0, 1, 2$ .

In Table 4 we will list in detail the 9 cases with  $t \equiv 0 \pmod{3}$  and, for each of them, in Table 5 we give the corresponding graph.

	$i$	$iii_1$	$iii_2$
$a_1$	$\emptyset$	$\emptyset$	$iii_{22}$
$a_2$	$\emptyset$	$iii_{13}$	$iii_{23}$
$a_3$	$\emptyset$	$iii_{12}$	$\emptyset$

	$i$	$iii_1$	$iii_2$
$a_4$	$i_2$	$\emptyset$	$\emptyset$
$a_5$	$i_2$	$iii_{13}$	$iii_{22}$
$a_6$	$i_2$	$iii_{12}$	$iii_{23}$

	$i$	$iii_1$	$iii_2$
$a_7$	$i_3$	$\emptyset$	$iii_{23}$
$a_8$	$i_3$	$iii_{13}$	$\emptyset$
$a_9$	$i_3$	$iii_{12}$	$iii_{22}$

Table 4:  $t \equiv 0 \pmod{3}$ .

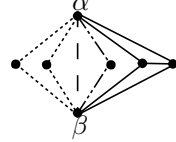
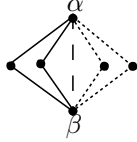
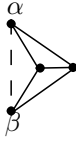


Figure 1: Case  $a_1$ .      Figure 2: Case  $a_8$ .      Figure 3: Cases  $a_5$  and  $a_9$ .

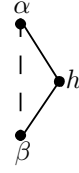
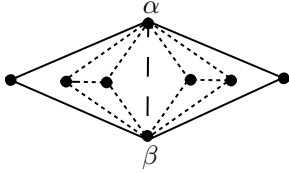


Figure 4: Cases  $a_2$ ,  $a_6$  and  $a_7$ .      Figure 5: Cases  $a_3$  and  $a_4$ .

Table 5: Graphs of the remnants plus edge  $[\alpha, \beta]$ .

It is easy to determine a  $P_4$ -decomposition of the graphs in Figures 1, 2, 3, 4. In cases  $a_3$  and  $a_4$  (Figure 5) a  $P_4$ -decomposition is clearly not possible, thus we proceed back tracking one step in the construction. How to deal with case  $a_3$  is explained in Figure 6.

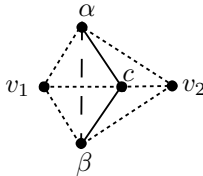


Figure 6: We recover the two  $P_4$ 's from 2 radii of the star of center  $c$ .

In case  $a_4$  we have to distinguish several subcases depending on the size of  $\mathcal{I}, \mathcal{C}$  and  $\mathcal{S}$ . When  $|\mathcal{I}| > 1$  see Figure 7. For  $|\mathcal{I}| = 1$  and  $|\mathcal{C}| \neq 0$ , see Figure 8.

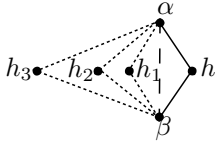


Figure 7: If  $|\mathcal{I}| > 1$ , we recover the two  $P_4$ 's from 3 vertices of  $\mathcal{I}$ .

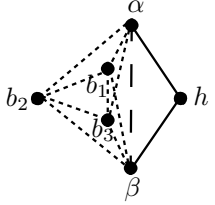


Figure 8: If  $|\mathcal{I}| = 1$  and  $|\mathcal{C}| \neq 0$  we recover the three  $P_4$ 's from a  $C_3$ .

When  $|\mathcal{I}| = 1$  and  $|\mathcal{C}| = 0$  we have two possibilities. If there is one star of  $\mathcal{S}$  with at least two edges, we proceed as explained in Figure 9.

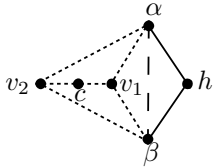


Figure 9: If  $|\mathcal{I}| = 1$ ,  $|\mathcal{C}| = 0$ , and  $\exists S_c \in \mathcal{S}$  with  $P_3 \leq S_c$  we recover the two  $P_4$ 's from 2 radii of  $S_c$ .

Otherwise,  $G''$  consists of an isolated vertex  $h$  and a set  $\mathcal{P}$  of disjoint  $P_2$ 's. Since  $|E(\Gamma')| \equiv 0 \pmod{3}$ , the size of  $\mathcal{P}$  is also divisible by 3, let  $|\mathcal{P}| = 3p$ . It is easy to see that for any 3-subset of  $\mathcal{P}$ , say  $P^3$ , the graph  $K_{A, P^3}$  has a  $P_4$ -decomposition. After  $p-1$  steps, the remnant is the graph in Figure 10, which likewise admits a  $P_4$ -decomposition. This concludes the case  $t \equiv 0 \pmod{3}$ .

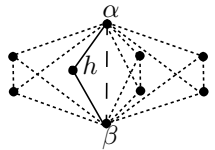


Figure 10: If  $|\mathcal{I}| = 1$ ,  $|\mathcal{C}| = 0$  and  $S$  is a disjoint union of  $P_2$ 's we recover the 15 edges from the last but one step.

With similar arguments, when  $t \equiv 1, 2 \pmod{3}$  it is possible to find a  $P_4$ -decomposition of  $E(\Gamma)$  leaving as remnants, respectively, one or two edges.  $\square$

### 3 Embeddings and down-links to $P_4$ -designs

The results presented in the previous section are used to prove the existence of embeddings and down-links to path designs. In particular, we shall focus our attention on  $P_4$ -decompositions.

**Theorem 3.1.** *Any partial  $(K_v, P_4)$ -design can be embedded into a  $(K_n, P_4)$ -design for any admissible  $n \geq v + 2$ .*

*Proof.* Let  $\mathcal{B}$  be a partial  $(K_v, P_4)$ -design. Let  $A$  be a set of vertices disjoint from  $V(K_v)$  with  $v + |A| \equiv 0, 1 \pmod{3}$  and  $|A| \geq 2$ . Let  $\Gamma$  be the graph such that  $V(\Gamma) = V(K_v) \cup A$  and  $E(\Gamma) = E(K_{v+|A|}) \setminus E(\mathcal{B})$ . Since  $|A| \geq 2$ , by Theorem 2.2 there exists a  $(\Gamma, P_4)$ -design  $\mathcal{B}'$  and, clearly,  $\mathcal{B} \cup \mathcal{B}'$  is a  $(K_{v+|A|}, P_4)$ -design.  $\square$

**Corollary 3.2.** *For any  $(K_v, \Gamma)$ -design with  $P_4 \leq \Gamma$*

$$\{n \geq v + 2 \mid n \equiv 0, 1 \pmod{3}\} \subseteq \mathcal{L}_2\Gamma(v) \subseteq \mathcal{L}_1\Gamma(v).$$

*Proof.* Let  $\mathcal{B}$  be a  $(K_v, \Gamma)$ -design with  $P_4 \leq \Gamma$ . Choose a  $P_4$  in each block of  $\mathcal{B}$  and call  $\mathcal{P}$  the set of such  $P_4$ 's. Obviously,  $\mathcal{P}$  is a partial  $P_4$ -decomposition of  $K_v$ . Hence, by Theorem 3.1,  $\mathcal{P}$  can be embedded into a  $(K_n, P_4)$ -design  $\mathcal{B}'$  for any admissible  $n \geq v + 2$ . The construction also guarantees the existence of a down-link from  $\mathcal{B}$  to  $\mathcal{B}'$ .  $\square$

**Theorem 3.3.** *For any even integer  $k$ , a  $P_k$ -design of order  $n \equiv 0, 1 \pmod{k-1}$  can be embedded into a  $P_k$ -design of any order  $m > n + 1$  with  $m \equiv 0, 1 \pmod{k-1}$ .*

*Proof.* Let  $\mathcal{B}$  be a  $(K_n, P_k)$ -design with  $n \equiv 0, 1 \pmod{k-1}$  and let  $m = n + s \equiv 0, 1 \pmod{k-1}$ . As  $K_{n+s} = K_n \cup K_s \cup K_{n,s}$ , for the existence of a  $(K_m, P_k)$ -design embedding  $\mathcal{B}$  it is enough to find a  $P_k$ -decomposition of  $K_s \cup K_{n,s}$ . Since  $n, n + s \equiv 0, 1 \pmod{k-1}$ , one of the following cases occurs

- $n = \lambda(k-1), s = \mu(k-1) \Rightarrow K_s \cup K_{n,s} = K_s \cup \lambda\mu K_{k-1, k-1}$
- $n = \lambda(k-1), s = 1 + \mu(k-1) \Rightarrow K_s \cup K_{n,s} = K_s \cup K_{\lambda(k-1), k + (\mu-1)(k-1)} = K_s \cup \lambda K_{k-1, k} \cup \lambda(\mu-1) K_{k-1, k-1}$
- $n = 1 + \lambda(k-1), s = \mu(k-1) \Rightarrow K_s \cup K_{n,s} = K_s \cup K_{k + (\lambda-1)(k-1), \mu(k-1)} = K_s \cup \mu K_{k, k-1} \cup \mu(\lambda-1) K_{k-1, k-1}$
- $n = 1 + \lambda(k-1), s = k-2 + \mu(k-1) \Rightarrow K_s \cup K_{n,s} = K_s \cup K_{1 + \lambda(k-1), s} = K_s \cup K_{1, s} \cup K_{\lambda(k-1), s} = K_{s+1} \cup K_{\lambda(k-1), k-2 + \mu(k-1)} = K_{s+1} \cup \lambda K_{k-1, k-2} \cup \lambda\mu K_{k-1, k-1}$

So, to find a  $P_k$ -decomposition of  $K_s \cup K_{n,s}$  it is sufficient to know  $P_k$ -decompositions of

- $K_s$  and  $K_{s+1}$ , which exist by [13],
- $K_{k-1,k-1}$ , whose existence is proved in [10],
- $K_{k-1,k}$  and  $K_{k-1,k-2}$ , whose existence follows from Proposition 2.1.

□

The following corollary is a straightforward consequence of Theorem 3.3.

**Corollary 3.4.** *If  $n \in \mathcal{L}_i\Gamma(v)$ , then*

$$\{m \geq n + 2 \mid m \equiv 0, 1 \pmod{3}\} \subseteq \mathcal{L}_i\Gamma(v).$$

**Remark 3.5.** *Set  $\eta_i = \inf \mathcal{L}_i\Gamma(v)$ . By Corollary 3.4,  $\mathcal{L}_i\Gamma(v)$  contains all admissible values  $m \geq \eta_i$  apart from (possibly)  $\eta_i + 1$ . Thus to exactly determine the spectra it is enough to compute  $\eta_i$  and ascertain if  $\eta_i + 1 \in \mathcal{L}_i\Gamma(v)$ .*

## 4 Cycle systems and path-designs

Here we shall provide some partial results on the existence of down-links from cycle systems and path-designs to  $P_4$ -designs.

We recall that a  $k$ -cycle system of order  $v$ , that is a  $(K_v, C_k)$ -design, exists if, and only if,  $k \leq v$ ,  $v$  is odd and  $v(v-1) \equiv 0 \pmod{2k}$ ; see [2], [12].

**Theorem 4.1.** *For any admissible  $v$  and any  $k \geq 9$*

$$\left\{ n \geq v - \left\lfloor \frac{k-9}{4} \right\rfloor \mid n \equiv 0, 1 \pmod{3} \right\} \subseteq \mathcal{L}_2 C_k(v) \subseteq \mathcal{L}_1 C_k(v).$$

*Proof.* Let  $k \geq 9$  and let  $\mathcal{B}$  be a  $(K_v, C_k)$ -design. Write  $t = \left\lfloor \frac{k-9}{4} \right\rfloor$ . Take  $t+2$  distinct vertices  $x_1, x_2, \dots, x_t, y_1, y_2 \in V(K_v)$ . Observe that it is possible to extract from each block  $C \in \mathcal{B}$  a  $P_4$  whose vertices are different from  $x_1, x_2, \dots, x_t, y_1, y_2$ , as we are forbidding at most  $4(t+1) + 2 = 4t + 6 = k - 3$  edges from any  $k$ -cycle. Use these  $P_4$ 's for the down-link. Let  $S$  be the image of the down-link, considered as a subgraph of  $K_{v-t} = K_v \setminus \{x_1, \dots, x_t\}$  and remove the edges of  $S$  from  $K_{v-t}$  to obtain a new graph  $R$ . It remains to show that  $R$  admits a  $P_4$ -decomposition. Observe that  $|V(R)| = v - t$  and  $y_1, y_2$  are two vertices of  $R$  of degree  $v - t - 1$ . To apply Theorem 2.2 we have to distinguish some cases according to the congruence class modulo 3 of  $v - t$ .

If  $v - t \equiv 0 \pmod{3}$ , then  $|E(R)| \equiv 0 \pmod{3}$  so the existence of a  $(R, P_4)$ -design is guaranteed by Theorem 2.2. Furthermore, if we add a vertex to  $K_{v-t}$  we can apply Theorem 2.2 also to  $R' = R \cup K_{1,v-t}$  since  $|E(R')| \equiv 0 \pmod{3}$ . Hence there exist down-links from  $\mathcal{B}$  to  $(K_{v-t}, P_4)$ -designs and



to  $(K_{v-t+1}, P_4)$ -designs.

If  $v - t \equiv 1 \pmod{3}$ , then  $|E(R)| \equiv 0 \pmod{3}$ , hence by Theorem 2.2, there exists a  $(R, P_4)$ -design. So we determine down-links from  $\mathcal{B}$  to  $(K_{v-t}, P_4)$ -designs.

Finally, if  $v - t \equiv 2 \pmod{3}$ , it is sufficient to add either  $u = 1$  or  $u = 2$  vertices to  $K_{v-t}$  and then apply Theorem 2.2 to  $R'' = (K_{v-t} \cup K_u \cup K_{v-t,u}) \setminus S$  in order to down-link  $\mathcal{B}$  to  $(K_{v-t+1}, P_4)$ -designs or to  $(K_{v-t+2}, P_4)$ -designs, respectively. The statement follows from Remark 3.5.  $\square$

Arguing exactly as in the previous proof it is possible to prove the following result.

**Theorem 4.2.** *For any admissible  $v$  and any  $k \geq 12$*

$$\left\{ n \geq v - \left\lfloor \frac{k-12}{4} \right\rfloor \mid n \equiv 0, 1 \pmod{3} \right\} \subseteq \mathcal{L}_2 P_k(v) \subseteq \mathcal{L}_1 P_k(v).$$

## 4.1 Small cases

We shall now investigate in detail the spectrum problems for  $\Gamma = C_4$  and  $\Gamma = P_5$ . In order to obtain our results, we shall extensively use the method of *gluing of down-links*, introduced in [3]. We briefly recall the main idea: a down-link from a  $(K_v, \Gamma)$ -design to a  $(K_n, \Gamma')$ -design can be constructed as union of down-links between partitions of the domain and the codomain. To give designs suitable for the down-link, we will use difference families; here we recall some preliminaries, for a survey see [1]. Let  $\Gamma$  be a graph. A set  $\mathcal{F}$  of graphs isomorphic to  $\Gamma$  with vertices in  $\mathbb{Z}_v$  is called a  $(v, \Gamma, 1)$ -*difference family* (DF, for short) if the list  $\Delta\mathcal{F}$  of differences from  $\mathcal{F}$ , namely the list of all possible differences  $x - y$ , where  $(x, y)$  is an ordered pair of adjacent vertices of an element of  $\mathcal{F}$ , covers  $\mathbb{Z}_v \setminus \{0\}$  exactly once. In [6] it is proved that if  $\mathcal{F} = \{B_1, \dots, B_t\}$  is a  $(v, \Gamma, 1)$ -DF, then the collection of graphs  $\mathcal{B} = \{B_i + g \mid B_i \in \mathcal{F}, g \in \mathbb{Z}_v\}$  is a cyclic  $(K_v, \Gamma)$ -design.

**Lemma 4.3.** *For any  $v \equiv 1, 9 \pmod{24}$ ,  $v > 1$ , there exists a down-link from a  $(K_v, C_4)$ -design to a  $(K_v, P_4)$ -design. For any  $v \equiv 9, 17 \pmod{24}$  there exists a down-link from a  $(K_v, C_4)$ -design to a  $(K_{v+1}, P_4)$ -design.*

*Proof.* Take  $v = s + 24t \geq 9$ , with  $s = 1, 9, 17$ , and  $V(K_v) = \mathbb{Z}_v$ . Consider the set of 4-cycles

$$\mathcal{C} = \left\{ C^a = \left( 0, a, \frac{v+1}{2}, \frac{v-1}{8} + a \right) \mid a = 1, 2, \dots, \frac{v-1}{8} \right\}.$$

It is straightforward to check that

$$\Delta C^a = \pm \left\{ a, \frac{v+1}{2} - a, \frac{3v+5}{8} - a, \frac{v-1}{8} + a \right\}.$$

Hence  $\Delta\mathcal{C} = \mathbb{Z}_v \setminus \{0\}$ , so, by [6], the  $C^a$  are the  $\frac{v-1}{8}$  base blocks of a cyclic  $(K_v, C_4)$ -design. The development of each base block gives  $v$  different 4-cycles, from each of which we extract the edge obtained by developing  $[0, a]$ . The obtained  $P_4$ 's will be used to define a down-link in a natural way. The removed edges can be connected to complete the  $P_4$ -decomposition of  $K_v$  as follows: for each triple  $\{[0, a+1], [0, a+2], [0, a+3]\}$ , for  $a \equiv 1 \pmod{3}$  where  $a \in \{1, 2, \dots, \frac{v-1}{8}\}$ , consider the three developments and connect the edges  $\{[i+1, a+1+(i+1)], [i, a+2+i], [i, a+3+i]\}$  obtaining the paths  $(i+1, a+i+2, i, a+i+3)$ , with  $i \in \mathbb{Z}_v$ .

If  $v \equiv 1 \pmod{24}$ , we have the required  $P_4$ -decomposition.

If  $v \equiv 9 \pmod{24}$ , we have the required  $P_4$ -decomposition except for the development of  $[0, 1]$ . The  $v$  edges of such a development can be easily connected to give the  $v$ -cycle  $C = (0, 1, \dots, v-1)$ , which obviously admits a  $P_4$ -decomposition. So, for  $v \equiv 1, 9 \pmod{24}$ , there exists a down-link from a  $(K_v, C_4)$ -design to a  $(K_v, P_4)$ -design. Under the assumption  $v \equiv 9 \pmod{24}$ ,  $n = v+1$  is also admissible. In this case, add the vertex  $\alpha$  to  $V(K_v)$  to obtain a  $K_{v+1}$  supporting the codomain of the down-link. Actually, the star  $S_{[\alpha; V]}$  of center  $\alpha$  and external vertices the elements of  $V(K_v)$  has been added. Proceed as before till to the last but one step, namely do not decompose the  $v$ -cycle  $C$  obtained by developing  $[0, 1]$ . So it remains to determine a  $P_4$ -decomposition of the wheel  $W = C \cup S_{[\alpha; V]}$ . It is easy to see that  $W$  can be decomposed into  $3 + 8t$  copies of the graph  $W'$  in Figure 11, which evidently admits a  $P_4$ -decomposition.

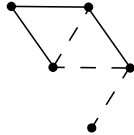


Figure 11: The graph  $W'$  as union of two  $P_4$ 's.

If  $v \equiv 17 \pmod{24}$ , proceeding as before, we determine the required  $P_4$ -decomposition except for the two developments, say  $d_1$  and  $d_2$ , of the edges  $[0, 1]$  and  $[0, \frac{v-1}{8}]$ . Keeping in mind that we must also add a vertex, say  $\alpha$ , to the codomain, we have to arrange the edges of  $d_1$ ,  $d_2$  and  $S_{[\alpha; V]}$ . It is easy to see that we can obtain the  $P_4$ 's as  $[\alpha, 1+i, i, \frac{v-1}{8}+i]$ , for  $i \in \mathbb{Z}_v$ .

So, for  $v \equiv 9, 17 \pmod{24}$  there exists a down-link from a  $(K_v, C_4)$ -design to a  $(K_{v+1}, P_4)$ -design.  $\square$

**Theorem 4.4.** *For any admissible  $v > 1$ ,*

$$\mathcal{L}_1 C_4(v) = \{n \geq v \mid n \equiv 0, 1 \pmod{3}\}; \quad (1)$$

$$\{n \geq v + 2 \mid n \equiv 0, 1 \pmod{3}\} \subseteq \mathcal{L}_2 C_4(v) \subseteq \{n \geq v \mid n \equiv 0, 1 \pmod{3}\}. \quad (2)$$

*Proof.* Let  $\mathcal{B}$  and  $\mathcal{B}'$  be, respectively, a  $(K_v, C_4)$ -design and a  $(K_n, P_4)$ -design. Suppose that  $\mathcal{B}$  can be down-linked to  $\mathcal{B}'$ . Clearly,  $n \geq v$ . Hence  $\mathcal{L}_2 C_4(v) \subseteq \mathcal{L}_1 C_4(v) \subseteq \{n \geq v \mid n \equiv 0, 1 \pmod{3}\}$ .

To prove the reverse inclusion in (1) observe that a  $(K_v, C_4)$ -design exists if, and only if,  $v \equiv 1 \pmod{8}$  and a  $(K_n, P_4)$ -design exists if, and only if,  $n \equiv 0, 1 \pmod{3}$ . So it makes sense to look for a down-link from a  $(K_v, C_4)$ -design to a  $(K_v, P_4)$ -design only for  $v \equiv 1, 9 \pmod{24}$ . Likewise, a down-link from a  $(K_v, C_4)$ -design to a  $(K_{v+1}, P_4)$ -design can exist only if  $v \equiv 9, 17 \pmod{24}$ . The existence of such down-links is proved in Lemma 4.3. The statement of (1) follows from Remark 3.5. The other inclusion in (2) immediately follows from Corollary 3.2.  $\square$

**Theorem 4.5.** *For any admissible  $v > 1$ ,*

$$\mathcal{L}_1 P_5(v) = \{n \geq v - 1 \mid n \equiv 0, 1 \pmod{3}\}; \quad (3)$$

$$\{n \geq v + 2 \mid n \equiv 0, 1 \pmod{3}\} \subseteq \mathcal{L}_2 P_5(v) \subseteq \{n \geq v \mid n \equiv 0, 1 \pmod{3}\}. \quad (4)$$

*Proof.* The first inclusion in (4) follows from Corollary 3.2. In order to prove the second, it is sufficient to show that for any admissible  $v$  there exists a  $(K_v, P_5)$ -design  $\mathcal{B}$  wherein no vertices can be deleted. In particular, this is the case if each vertex of  $K_v$  has degree 2 in at least one block of  $\mathcal{B}$ . First of all note that in a  $(K_v, P_5)$ -design there is at most one vertex with degree 1 in each block where it appears. Suppose that there actually exists a  $(K_v, P_5)$ -design  $\overline{\mathcal{B}}$  with a vertex  $x$  as above. It is easy to see that in  $\overline{\mathcal{B}}$  there is at least one block  $P^1 = [x, a, b, c, d]$  such that the vertices  $a, b$  and  $c$  have degree two in at least another block. Let  $P^2 = [x, d, e, f, g]$ . By reassembling the edges of  $P^1 \cup P^2$ , it is possible to replace in  $\overline{\mathcal{B}}$  these two paths with  $P^3 = [d, x, a, b, c]$ ,  $P^4 = [c, d, e, f, g]$  if  $c \neq f, g$  or  $P^5 = [a, x, d, c, g]$ ,  $P^6 = [a, b, c, e, d]$  if  $c = f$  or  $P^7 = [c, d, x, a, b]$ ,  $P^8 = [b, c, f, e, d]$  if  $c = g$ . Thus we have again a  $(K_v, P_5)$ -design. By the assumption on  $a, b, c$  all the vertices of this new design have degree two in at least one block.

Now we consider Relation (3). Let  $\mathcal{B}$  and  $\mathcal{B}'$  be respectively a  $(K_v, P_5)$ -design and a  $(K_n, P_4)$ -design. Suppose there exists a down-link  $f: \mathcal{B} \rightarrow \mathcal{B}'$ . Clearly,  $n > v - 2$ . Hence,  $\mathcal{L}_1 P_5(v) \subseteq \{n \geq v - 1 \mid n \equiv 0, 1 \pmod{3}\}$ .

To show the reverse inclusion in (3) we prove the actual existence of designs providing down-links. Since a  $(K_v, P_5)$ -design exists if, and only if,  $v \equiv 0, 1 \pmod{8}$  and a  $(K_n, P_4)$ -design exists if, and only if,  $n \equiv 0, 1 \pmod{3}$ , it makes sense to look for a down-link from a  $(K_v, P_5)$ -design to a  $(K_{v-1}, P_4)$ -design only if  $v \equiv 1, 8, 16, 17 \pmod{24}$ . For the same reason, it makes sense

to construct a down-link from a  $(K_v, P_5)$ -design to a  $(K_v, P_4)$ -design only for  $v \equiv 0, 1, 9, 16 \pmod{24}$ . In view of Remark 3.5, in order to complete the proof, we have also to provide a down-link from a  $(K_v, P_5)$ -design to a  $(K_{v+1}, P_4)$ -design for every  $v \equiv 0, 9 \pmod{24}$ .

To determine the necessary down-links, we analyze a few basic cases and then apply the *gluing method*. To this end, we will use the following obvious relations in an appropriate way:  $K_{a+b} = K_a \cup K_b \cup K_{a,b}$  and  $K_{a+b,c} = K_{a,c} \cup K_{b,c}$ . In particular,

$$\begin{aligned} K_{\ell+24t} &= K_\ell \cup K_{24t} \cup K_{\ell,24t}; \\ K_{24t} &= tK_{24} \cup \binom{t}{2}K_{24,24} = tK_{24} \cup 48\binom{t}{2}K_{3,4}; \\ K_{\ell=rs,24t} &= rK_{s,24t} = rtK_{s,24} = 6rtK_{s,4} = 8rtK_{s,3}. \end{aligned}$$

Let us now examine the possible cases.

- $(K_v, P_5) \rightarrow (K_{v-1}, P_4)$ -design with  $v = \ell + 24t > 1$ ,  $\ell = 1, 8, 16, 17$ .

$P_5$ -design of order	basic components	$\rightarrow$	basic components	$P_4$ -design of order
$1 + 24t$	$(K_{25}, P_5), (K_{3,4}, P_5)$		$(K_{24}, P_4), (K_{3,4}, P_4)$	$24t$
$8 + 24t$	$(K_8, P_5), (K_{24}, P_5)$ $(K_{4,3}, P_5)$		$(K_7, P_4), (K_{24}, P_4)$ $(K_{4,3}, P_4), (K_{3,3}, P_4)$	$7 + 24t$
$16 + 24t$	$(K_{16}, P_5), (K_{24}, P_5)$ $(K_{4,3}, P_5)$		$(K_{15}, P_4), (K_{24}, P_4)$ $(K_{4,3}, P_4), (K_{3,3}, P_4)$	$15 + 24t$
$17 + 24t$	$(K_{17}, P_5), (K_{24}, P_5)$ $(K_{4,3}, P_5)$		$(K_{16}, P_4), (K_{24}, P_4)$ $(K_{4,3}, P_4), (K_{3,3}, P_4)$	$16 + 24t$

- $(K_v, P_5) \rightarrow (K_v, P_4)$ -design with  $v = \ell + 24t > 1$ ,  $\ell = 0, 1, 9, 16$ .

$P_5$ -design of order	basic components	$\rightarrow$	basic components	$P_4$ -design of order
$24t$	$(K_{24}, P_5), (K_{3,4}, P_5)$		$(K_{24}, P_4), (K_{3,4}, P_4)$	$24t$
$1 + 24t$	$(K_9, P_5), (K_{16}, P_5)$ $(K_{24}, P_5), (K_{3,4}, P_5)$		$(K_9, P_4), (K_{16}, P_4)$ $(K_{24}, P_4), (K_{3,4}, P_4)$	$1 + 24t$
$9 + 24t$	$(K_9, P_5), (K_{24}, P_5)$ $(K_{3,4}, P_5)$		$(K_9, P_4), (K_{24}, P_4)$ $(K_{3,4}, P_4)$	$9 + 24t$
$16 + 24t$	$(K_{16}, P_5), (K_{24}, P_5)$ $(K_{3,4}, P_5)$		$(K_{16}, P_4), (K_{24}, P_4)$ $(K_{3,4}, P_4)$	$16 + 24t$

- $(K_v, P_5) \rightarrow (K_{v+1}, P_4)$ -design with  $v = \ell + 24t > 1$ ,  $\ell = 0, 9$ .

$P_5$ -design of order	basic components	$\rightarrow$	basic components	$P_4$ -design of order
$24t$	$(K_{24}, P_5), (K_{3,4}, P_5)$		$(K_{25}, P_4), (K_{3,4}, P_4)$	$1 + 24t$
$9 + 24t$	$(K_9, P_5), (K_{24}, P_5)$ $(K_{3,4}, P_5), (K_{9,24}, P_5)$		$(K_{10}, P_4), (K_{24}, P_4)$ $(K_{3,4}, P_4), (K_{10,24}, P_4)$	$10 + 24t$

It is straightforward to show the existence of such basic down-links. For instance we provide a down-link  $\xi$  from a  $(K_{9,24}, P_5)$ -design to a  $(K_{10,24}, P_4)$ -design. Let  $A = \{a, b, c, d, e, f, g, h, i\}$  and  $B = \mathbb{Z}_{24}$ , that is  $K_{9,24} = K_{A,B}$ . The following are the 54 paths of a  $P_5$ -decomposition of  $K_{A,B}$ :

<u>[6, a, 12, b, 1]</u>	<u>[1, c, 12, d, 6]</u>	<u>[6, e, 18, f, 1]</u>	<u>[1, g, 12, h, 0]</u>	<u>[12, i, 0, a, 18]</u>
<u>[7, a, 13, b, 2]</u>	<u>[2, c, 13, d, 7]</u>	<u>[7, e, 19, f, 2]</u>	<u>[2, g, 13, h, 1]</u>	<u>[13, i, 1, a, 19]</u>
<u>[8, a, 14, b, 3]</u>	<u>[3, c, 14, d, 8]</u>	<u>[8, e, 20, f, 3]</u>	<u>[3, g, 14, h, 2]</u>	<u>[14, i, 2, a, 20]</u>
<u>[9, a, 15, b, 4]</u>	<u>[4, c, 15, d, 9]</u>	<u>[9, e, 21, f, 4]</u>	<u>[4, g, 15, h, 3]</u>	<u>[15, i, 3, a, 21]</u>
<u>[10, a, 16, b, 5]</u>	<u>[5, c, 16, d, 10]</u>	<u>[10, e, 22, f, 5]</u>	<u>[5, g, 16, h, 4]</u>	<u>[16, i, 4, a, 22]</u>
<u>[11, a, 17, b, 0]</u>	<u>[0, c, 17, d, 11]</u>	<u>[11, e, 23, f, 0]</u>	<u>[0, g, 17, h, 5]</u>	<u>[17, i, 5, a, 23]</u>
<u>[18, b, 6, c, 19]</u>	<u>[19, d, 0, e, 12]</u>	<u>[12, f, 6, g, 19]</u>	<u>[19, h, 6, i, 18]</u>	
<u>[19, b, 7, c, 20]</u>	<u>[20, d, 1, e, 13]</u>	<u>[13, f, 7, g, 20]</u>	<u>[20, h, 7, i, 19]</u>	
<u>[20, b, 8, c, 21]</u>	<u>[21, d, 2, e, 14]</u>	<u>[14, f, 8, g, 21]</u>	<u>[21, h, 8, i, 20]</u>	
<u>[21, b, 9, c, 22]</u>	<u>[22, d, 3, e, 15]</u>	<u>[15, f, 9, g, 22]</u>	<u>[22, h, 9, i, 21]</u>	
<u>[22, b, 10, c, 23]</u>	<u>[23, d, 4, e, 16]</u>	<u>[16, f, 10, g, 23]</u>	<u>[23, h, 10, i, 22]</u>	
<u>[23, b, 11, c, 18]</u>	<u>[18, d, 5, e, 17]</u>	<u>[17, f, 11, g, 18]</u>	<u>[18, h, 11, i, 23]</u>	

We obtain the image of any  $P_5$  via  $\xi$  by removing the underlined edge. Now, to complete the codomain, we have to add a further vertex to  $A$ , say  $\alpha$ , together with all the edges connecting  $\alpha$  to the vertices of  $B$ . Thus, it remains to decompose the graph formed by the removed edges together with the star of center  $\alpha$  and external vertices in  $B$ . Such a  $P_4$ -decomposition is listed below:

<u>[6, a, 9, <math>\alpha</math>]</u>	<u>[7, a, 10, <math>\alpha</math>]</u>	<u>[8, a, 11, <math>\alpha</math>]</u>	<u>[1, c, 4, <math>\alpha</math>]</u>	<u>[2, c, 5, <math>\alpha</math>]</u>	
<u>[3, c, 0, <math>\alpha</math>]</u>	<u>[9, e, 6, <math>\alpha</math>]</u>	<u>[10, e, 7, <math>\alpha</math>]</u>	<u>[11, e, 8, <math>\alpha</math>]</u>	<u>[4, g, 1, <math>\alpha</math>]</u>	
<u>[5, g, 2, <math>\alpha</math>]</u>	<u>[0, g, 3, <math>\alpha</math>]</u>	<u>[15, i, 12, <math>\alpha</math>]</u>	<u>[16, i, 13, <math>\alpha</math>]</u>	<u>[17, i, 14, <math>\alpha</math>]</u>	
<u>[22, d, 19, <math>\alpha</math>]</u>	<u>[23, d, 20, <math>\alpha</math>]</u>	<u>[21, d, 18, <math>\alpha</math>]</u>	<u>[12, f, 15, <math>\alpha</math>]</u>	<u>[13, f, 16, <math>\alpha</math>]</u>	
<u>[14, f, 17, <math>\alpha</math>]</u>	<u>[20, h, 21, <math>\alpha</math>]</u>	<u>[23, h, 22, <math>\alpha</math>]</u>	<u>[20, b, 23, <math>\alpha</math>]</u>	<u>[h, 18, b, 21]</u>	<u>[h, 19, b, 22]</u> .

□

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